New Nonsingular Forms of Perturbed Satellite Equations of Motion

Ahmed Aly Kamel*
Ford Aerospace and Communication Corporation, Palo Alto, California

Vector calculus is used to derive two new forms of the equations of motion of perturbed satellites. Both forms are given in terms of nonsingular orbital elements in extended vector space that eliminate the ambiguity in the variation of retrograde orbits. In the first form, the orbital eccentricity, true anomaly, argument of latitude, and orbital inclination are substituted by four fast variables whose unperturbed periods are essentially equal to the orbital period. In the second form, these four fast variables are transformed to four slow variables such that the eccentricity vector variation is independent of the out-of-plane acceleration. The transformation matrix between these new nonsingular elements and Cartesian coordinates is given and perturbing accelerations due to the Earth and a third body are developed. Applications of the perturbation methods of multiple scale and averagings to the developed equations of motion are demonstrated.

I. Introduction

THE satellite equations of motion can be expressed in various forms. A simple Cartesian form[1,2] is widely used in connection with numerical integration. More sophisticated forms based on the variation of parameters developed by Lagrange[3] are particularly useful in connection with perturbation theories. Lagrange's equations can be transformed to canonical forms[3-5] or nonsingular noncanonical forms.[6,7] The independent variable is usually the time but in some cases, e.g., Earth's zonal harmonic perturbations, it is advantageous to use an independent variable other than time[8-10] (e.g., true anomaly or a similar fast angle).

The purpose of this paper is to develop two new forms of Lagrange's equations of motion [Eqs. (12) and (19)] in terms of nonsingular orbital elements whose variations are determined by the components (F_r, F_θ, F_h) of a perturbing acceleration in the Euler-Hill rotating frame depicted in Fig. 1.

Section II develops the equations of motion in terms of the fast variables [Eq. (12)]. This form is particularly useful for the application of perturbation method of Lindested-Poincaré and the method of multiple scale.[11-13] Section III derives the equations of motion in terms of slow variables [Eq. (19)] in a form suitable for the application of the method of averaging,[14] Lie transform,[15,16] and Lie series.[17] Section IV summarizes the transformation between the Cartesian and the nonsingular orbital elements useful for the evaluation of the initial conditions. Section V develops the form of the perturbing acceleration of the Earth and a third body in terms of the obtained variables. Sections VI and VII demonstrate the application of the method of multiple scale and the methods of averaging to the obtained equations of motion.

II. Equations of Motion in Terms of Fast Variables

This section starts with the equations of motion in the Cartesian form and applies vector calculus to obtain the equations of motion in terms of a set of fast nonsingular orbital elements.

Following Newton's law of gravitation, the equations of motion of a satellite can be written in the vectorial form

$$\dot{r} = v \qquad \dot{v} = -\frac{\mu_e}{r^3} r + f(r, v, t) \tag{1}$$

where

- r = satellite position (radius) vector pointing from the center of the Earth to the satellite
- r = satellite radius = magnitude of the position vector
- v = satellite velocity vector
- μ_e = Earth's gravitational constant
- f = perturbing acceleration vector, and the dot over a vector denotes its differentiation with respect to the time t

Now, define the angular momentum vector h and the eccentricity vector e (Eqs. 1.4-3 and 1.5-10 of Ref. 18) as follows

$$h = r \times v \qquad e = 1/\mu_e v \times h - r/r \tag{2}$$

where the $\times$ denotes the cross product between two vectors.

At this point, a coordinate system has to be selected to resolve various vectors into components. Figure 1 shows two well-known systems. The first is the inertially fixed geocentric coordinate system defined by the unit vectors $(\hat{x}, \hat{y}, \hat{z})$ with $\hat{x}$ along the vernal equinox at epoch, $\hat{z}$ along the north pole, and $\hat{y}$ determined by the right-hand rule. The second is the Euler-Hill rotating coordinate system defined by Ω (right ascension of ascending node), I (orbital inclination), and θ (argument of latitude). The rotating unit vectors $(\hat{r}, \hat{\theta}, \hat{h})$ have $\hat{r}$ along the radius vector and $\hat{\theta}$ along the transverse component of the velocity vector such that $(\hat{r}, \hat{\theta})$ are in the plane containing the instantaneous satellite radius and velocity (osculating orbit) and $\hat{h}$ is along the osculating orbit normal (angular momentum vector direction).

In view of Fig. 1 (see also Eq. 11.147 of Ref. 19), the relationship between $(\hat{x}, \hat{y}, \hat{z})$ and $(\hat{r}, \hat{\theta}, \hat{h})$ are defined by the orthogonal matrix M as follows

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = M \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{bmatrix}, \qquad M^{-1} = M^T \tag{3}$$

where

$$M = \begin{bmatrix} c_\Omega c_\theta - c_I s_\Omega s_\theta & -c_\Omega s_\theta - c_I s_\Omega c_\theta & s_I s_\Omega \\ s_\Omega c_\theta + c_I c_\Omega s_\theta & -s_\Omega s_\theta + c_I c_\Omega c_\theta & -s_I c_\Omega \\ s_I s_\theta & s_I c_\theta & c_I \end{bmatrix}$$

$$c_\sigma = \cos\sigma \qquad s_\sigma = \sin\sigma \qquad \sigma = (\Omega, \theta, I)$$

Presented as Paper 82-0074 at the AIAA 20th Aerospace Sciences Meeting, Orlando, Fla., Jan. 11-14, 1982; submitted Sept. 21, 1982; revision received Nov. 3, 1982.

*Supervisor, Orbit Dynamics Section, Mission Analysis Department.

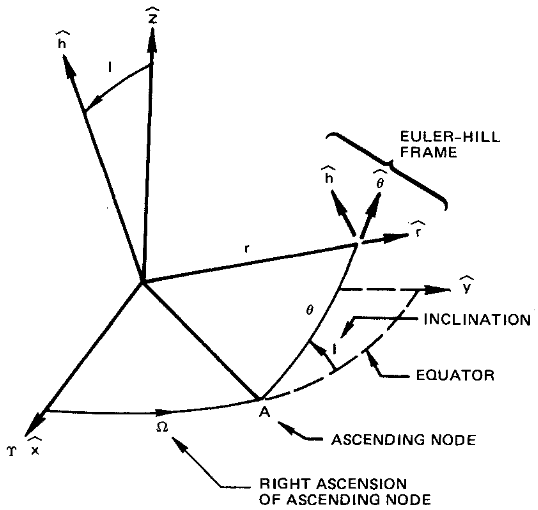

Fig. 1 Geometry of the inertial and Euler-Hill frames.

Also, the angular velocity vector ω of the rotating Euler-Hill system $(\hat{r},\hat{\theta},\hat{h})$ relative to the inertially fixed system $(\hat{x},\hat{y},\hat{z})$ is given by

$$\omega=\omega_r\hat{r}+\omega_\theta\hat{\theta}+\omega_h\hat{h} \tag{4}$$

In view of Fig. 1 (see also Eq. 4.78 of Ref. 19), we get

$$\omega_r=\dot{\Omega}s_Is_\theta+\dot{I}c_\theta,\qquad \omega_\theta=\dot{\Omega}s_Ic_\theta-\dot{I}s_\theta,\qquad \omega_h=\dot{\theta}+\dot{\Omega}c_I \tag{5}$$

The differentiation of the unit vectors $(\hat{x},\hat{y},\hat{z})$ and $(\hat{r},\hat{\theta},\hat{h})$ with respect to time are given by

$$\dot{\hat{x}}=\dot{\hat{y}}=\dot{\hat{z}}=0$$

$$\dot{\hat{r}}=\omega\times\hat{r},\qquad \dot{\hat{\theta}}=\omega\times\hat{\theta},\qquad \dot{\hat{h}}=\omega\times\hat{h} \tag{6}$$

The vectors $\boldsymbol{r}$, $\boldsymbol{v}$, $\boldsymbol{f}$, $\boldsymbol{h}$, and $\boldsymbol{e}$ can now be expressed in terms of components in the Euler-Hill frame as follows

$$\boldsymbol{r}=r\hat{r},\qquad \boldsymbol{v}=v_r\hat{r}+v_\theta\hat{\theta},\qquad \boldsymbol{h}=h\hat{h},\qquad \boldsymbol{e}=e_r\hat{r}+e_\theta\hat{\theta}$$

$$\boldsymbol{f}=f_r\hat{r}+f_\theta\hat{i}+f_h\hat{h} \tag{7}$$

Also, introducing the following auxiliary variables $(\ell,\ q_1,\ q_2,\ q_3)$ defined by

$$\ell=\Omega+\theta,\qquad \hat{z}=q_1\hat{r}+q_2\hat{\theta}+q_3\hat{h} \tag{8}$$

In view of Eqs. (3) and (8), we get

$$q_1=s_Is_\theta,\qquad q_2=s_Ic_\theta,\qquad q_3=c_I \tag{9}$$

Now, differentiations of $(\boldsymbol{r},\boldsymbol{v},\boldsymbol{e},\boldsymbol{h},\hat{z})$ of Eqs. (2), (7), and (8) with respect to time and using Eqs. (1), (4), (6), (7), and (9) lead to

$$\omega_r=\frac{rf_h}{h},\qquad \omega_\theta=0,\qquad \omega_h=\frac{h}{r^2} \tag{10}$$

$$r=\frac{h^2/\mu_e}{1+e_r},\qquad v_r=-\frac{\mu_e}{h}e_\theta,\qquad \varphi_\theta=\frac{h}{r} \tag{11}$$

plus the following equations of motion [Eqs. (5) and (10) are used for δ' and t' computation]

$$\begin{aligned}
&h'=hF_\theta,\qquad e_r'=e_\theta+2(1+e_r)F_\theta\\
&e_\theta'=-e_r-(1+e_r)F_r+e_\theta F_\theta\\
&q_1'=q_2,\qquad q_2'=-q_1+q_3F_h,\qquad q_3'=-q_2F_h\\
&\delta'=\alpha_1F_h,\qquad t'=r^2/h
\end{aligned} \tag{12}$$

where

$$(F_r,F_\theta,F_h)=(r^3/h^2)(f_r,f_\theta,f_h),\quad \alpha_1=q_1/(1+q_3),\quad \delta=\ell-\beta$$

and the prime over a variable means its differentiation with respect to the angle β defined by

$$\dot{\beta}=\omega_h=h/r^2 \tag{13}$$

Note that Eq. (12) contains redundant information due to the fact

$$q_1^2+q_2^2+q_3^2=1$$

$$\beta(t=0)=\beta_0=\text{arbitrary value} \tag{14}$$

Equation (12) can therefore be reduced to six variables by eliminating q_3 using Eq. (14) and by changing the independent variable β to the time t using Eq. (13). The use of the additional variable q_3, however, eliminates the need for square root computation and, more importantly, avoids the ambiguity in case of retrograde orbits.

It is important to point out that the above equations of motion and the associated transformations are also applicable to the case when the orbital inclination I is in the neighborhood of 180 deg (i.e., $q_3\cong-1$). In this case, however, (ℓ,q_2,q_2,q_3) of Eqs. (8) and (9) are redefined as

$$(\ell,q_1,q_2,q_3)=-(\Omega-\theta,s_Is_\theta,s_Ic_\theta,c_I) \tag{15}$$

and the Cartesian coordinate system is redefined to have $\hat{x}$ along the vernal equinox, $\hat{z}$ along the Earth's south pole (instead of north pole), and $\hat{y}$ as determined by the right-hand rule.

It should also be pointed out that the true anomaly f and the argument of perigee ω do not play any separate roles in the above development. They can be obtained, however, from the following equations

$$e_r=e\cos f,\qquad e_\theta=-e\sin f,\qquad \omega=\theta-f \tag{16}$$

where e is the orbital eccentricity.

III. Transformation to Slow Variables

Equation (12) can be transformed to a slowly varying form using the equinoctial elements used by Cefola[6] and by Nacozy-Dallas.[7] In the following development, a new set (e_1,e_2,h_1,h_2) that has the advantage of decoupling the in-plane from the out-of-plane direct effects will be used. This set is defined by the rotational equations

$$\begin{aligned}
&e_r=e_1c_\beta+e_2s_\beta,\quad e_\theta=-e_1s_\beta+e_2c_\beta\\
&q_1=h_1s_\beta-h_2c_\beta,\quad q_2=h_1c_\beta+h_2s_\beta,\quad q_3=h_3
\end{aligned} \tag{17}$$

The use of Eqs. (8), (9), and (16) in Eq. (17) leads to

$$\begin{aligned}
&e_1=e_rc_\beta-e_\theta s_\beta=ec_{\omega+\gamma},\qquad e_2=e_rs_\beta+e_\theta c_\beta=es_{\omega+\gamma}\\
&h_1=q_1s_\beta+q_2c_\beta=s_Ic_\gamma,\qquad h_2=-q_1c_\beta+q_2s_\beta=s_Is_\gamma
\end{aligned} \tag{18}$$

where e is the eccentricity and $\gamma=\Omega-\delta$ is obtained using $\delta=\ell-\beta$.

Substitution of Eq. (18) in Eq. (12) leads to

$$\begin{aligned}
&h'=hF_\theta\\
&e_1'=[(2+e_r)c_\beta+e_1]F_\theta+(1+e_r)s_\beta F_r\\
&e_2'=[(2+e_r)s_\beta+e_2]F_\theta-(1+e_r)c_\beta F_r\\
&h_1'=h_3c_\beta F_h,\qquad h_2'=h_3s_\beta F_h,\qquad h_3'=-(h_1c_\beta+h_2s_\beta)F_h\\
&\delta'=\alpha_1F_h,\qquad t'=r^2/h
\end{aligned} \tag{19}$$

For orbital inclination I in the neighborhood of 180 deg (i.e., $h_3 \cong -1$), the (γ, h_1, h_2, h_3) are redefined using Eq. (15) as follows:

$$(\gamma, h_1, h_2, h_3) = -(\Omega+\delta, \quad s_I c_\gamma, \quad s_I s_\gamma, \quad c_I) \tag{20}$$

Equations (18) and (19) show that the nonsingular elements presented in this paper have some advantage over those referred to the equinox,[6,7] namely, the decoupling of the in-plane and out-of-plane direct effects. This is achieved by referring the orbital elements to an axis that is slowly rotating in the equatorial plane with the angle δ relative to the equinox. The initial value of the angle δ is arbitrary and, therefore, can be selected to be zero.

IV. Transformation from and to Cartesian Coordinates

The matrix M of Eq. (3) can be expressed in terms of $(\beta, \delta, q_1, q_2, q_3)$ by using Eq. (8), (9), or (15) or in terms of $(\beta, \delta, h_1, h_2, h_3)$ by using Eq. (17) or (20). This leads to

$$M = \begin{bmatrix} c_{\beta+\delta} + \alpha_1 m_1 & -s_{\beta+\delta} + \alpha_2 m_1 & m_1 \\ s_{\beta+\delta} + \alpha_1 m_2 & c_{\beta+\delta} + \alpha_2 m_2 & m_2 \\ q_1 & q_2 & q_3 \end{bmatrix} \tag{21}$$

where

$$\alpha_i = q_i/(1+q_3), \qquad i=1,2$$

$$m_1 = q_2 s_{\beta+\delta} - q_1 c_{\beta+\delta} = h_1 s_\delta + h_2 c_\delta$$

$$m_2 = -(q_1 s_{\beta+\delta} + q_2 c_{\beta+\delta}) = -h_1 c_\delta + h_2 s_\delta$$

Also, transformation between the components of $(\boldsymbol{r}, \boldsymbol{v}, \boldsymbol{h})$ in the Cartesian and Euler-Hill frames can be put in the form

$$\begin{bmatrix} x & \dot{x} & h_x \\ y & \dot{y} & h_y \\ z & \dot{z} & h_z \end{bmatrix} = M \begin{bmatrix} r & v_r & 0 \\ 0 & v_\theta & 0 \\ 0 & 0 & h \end{bmatrix} \tag{22}$$

where (r, v_r, v_θ, h) are given by Eq. (11). They can also be expressed in terms of the Cartesian components as follows

$$r = [x^2+y^2+z^2]^{1/2}, \qquad v_r = \frac{x\dot{x}+y\dot{y}+z\dot{z}}{r}$$

$$h = [h_x^2+h_y^2+h_z^2]^{1/2}, \qquad h_x = y\dot{z} - z\dot{y}$$

$$h_y = z\dot{x} - x\dot{z}, \qquad h_z = x\dot{y} - y\dot{x}$$

Equations (21) and (22) can be used to evaluate initial conditions for Eq. (12) or (19) assuming $\delta = 0$. These same equations can also be used to obtain the Cartesian components of $\boldsymbol{r}$ and $\boldsymbol{v}$ once either Eq. (12) or (19) is integrated.

Finally, it should be mentioned that the matrix M can be put in the form

$$M = RM_1 \tag{23}$$

where

$$R = \begin{bmatrix} c_\delta & -s_\delta & 0 \\ s_\delta & c_\delta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad M_1 = M|_{\delta=0}$$

V. Perturbing Accelerations

This section derives the perturbing accelerations of the Earth and a third body in the form needed for the integration of Eq. (12) or (19).

Following Pines[20] and Broucke,[21] the Earth's perturbing potential can be put in the form

$$U_e = U_e\left(r, \frac{x_e}{r}, \frac{y_e}{r}, \frac{z_e}{r}\right) \tag{24}$$

where x_e, y_e, and z_e are the components of the position vector $\boldsymbol{r}$ in the earth-fixed coordinate system.

The acceleration $\boldsymbol{f}_e$ is the gradient of U_e

$$\boldsymbol{f}_e = \nabla U_e = \frac{\partial U_e}{\partial r}\nabla r + \frac{\partial U_e}{\partial\left(\frac{x_e}{r}\right)}\nabla\left(\frac{x_e}{r}\right) + \frac{\partial U_e}{\partial\left(\frac{y_e}{r}\right)}\nabla\left(\frac{y_e}{r}\right) + \frac{\partial U_e}{\partial\left(\frac{z_e}{r}\right)}\nabla\left(\frac{z_e}{r}\right) \tag{25}$$

Now, to obtain f_{re}, $f_{\theta e}$, and f_{he} of Eq. (12), the $\nabla(r, x_e/r, y_e/r, z_e/r)$ have to be expressed in terms of $(\hat{r}, \hat{\theta}, \hat{h})$ as follows:

$$\nabla r = \hat{r}, \qquad \nabla\frac{\xi}{r} = \frac{1}{r}\left[\hat{\xi} - \frac{\xi}{r}\hat{r}\right], \qquad \xi = (x_e, y_e, z_e) \tag{26}$$

In view of Eqs. (3) and (21), we get

$$\begin{bmatrix} \hat{x}_e \\ \hat{y}_e \\ \hat{z}_e \end{bmatrix} = M_e \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{bmatrix} \tag{27}$$

where M_e has the same form as M of Eq. (21) with β replaced by $\beta - \mathrm{GHA}$, where GHA is the Greenwich hour angle.

Substitution of Eq. (27) into Eq. (26) leads to

$$\nabla r = \hat{r}$$

$$r\nabla\left(\frac{x_e}{r}\right) = -[(1-\alpha_2 q_2)s_{\ell_e} + \alpha_2 q_1 c_{\ell_e}]\hat{\theta} - [q_1 c_{\ell_e} - q_2 s_{\ell_e}]\hat{h}$$

$$r\nabla\left(\frac{y_e}{r}\right) = [(1-\alpha_2 q_2)c_{\ell_e} - \alpha_2 q_1 s_{\ell_e}]\hat{\theta} - [q_1 s_{\ell_e} + q_2 c_{\ell_e}]\hat{h}$$

$$r\nabla\left(\frac{z_e}{r}\right) = q_2\hat{\theta} + q_3\hat{h} \tag{28}$$

where

$$\ell_e = \beta + \delta - \mathrm{GHA}$$

Also, in view of Eqs. (3), (7), and (27)

$$x_e/r = (1-\alpha_1 q_1)c_{\ell_e} + \alpha_1 q_2 s_{\ell_e}$$

$$y_e/r = (1-\alpha_1 q_1)s_{\ell_e} - \alpha_1 q_2 c_{\ell_e}, \qquad z_e/r = q_1 \tag{29}$$

Substitution of Eq. (28) into Eq. (25) now leads to

$$f_{re} = \frac{\partial U_e}{\partial r}, \qquad rf_{\theta e} = \alpha_2 q_1 A_e - (1-\alpha_2 q_2)B_e + q_2 C_e$$

$$rf_{he} = q_1 A_e + q_2 B_e + q_3 C_e \tag{30}$$

where

$$A_e = -\frac{\partial U_e}{\partial\left(\frac{x_e}{r}\right)} c_{\ell_e} - \frac{\partial U_e}{\partial\left(\frac{y_e}{r}\right)} s_{\ell_e}$$

$$B_e = \frac{\partial U_e}{\partial\left(\frac{x_e}{r}\right)} s_{\ell_e} - \frac{\partial U_e}{\partial\left(\frac{y_e}{r}\right)} c_{\ell_e}$$

$$C_e = \frac{\partial U_e}{\partial\left(\frac{z_e}{r}\right)}$$

The functions (A_e, B_e, C_e) can be derived either in a recursive or a nonrecursive form in terms of $(h, e_r, q_1, q_2, q_3, \ell_e)$. This can also be transformed to a form in terms of $(h, e_1, e_2, h_1, h_2, h_3, \beta, \ell_e)$ using Eq. (17).

Following Brown[22] and Broucke,[23] the perturbing potential function of a third body can be put in a form similar to Eq. (24)

$$U_3 = U_3(r, x/r, y/r, z/r, t) \tag{31}$$

The addition of the time t is due to the third-body motion. Formulas for computer usage can be found in Ref. 24.

Now, following similar steps as in the Earth acceleration derivation leads to

$$rf_{r3} = \frac{\partial U_3}{\partial r}, \qquad rf_{\theta 3} = \alpha_2 q_1 A_3 - (1-\alpha_2 q_2) B_3 + q_2 C_3$$

$$rf_{h3} = q_1 A_3 + q_2 B_3 + q_3 C_3 \tag{32}$$

where

$$A_3 = -\frac{\partial U_3}{\partial\left(\frac{x}{r}\right)} c_{\beta+\delta} - \frac{\partial U_3}{\partial\left(\frac{y}{r}\right)} s_{\beta+\delta}$$

$$B_3 = \frac{\partial U_3}{\partial\left(\frac{x}{r}\right)} s_{\beta+\delta} - \frac{\partial U_3}{\partial\left(\frac{y}{r}\right)} c_{\beta+\delta}$$

$$C_3 = \frac{\partial U_3}{\partial\left(\frac{z}{r}\right)}$$

and x/r, y/r, and z/r are given by Eq. (22) as

$$x/r = (1-\alpha_1 q_1) c_{\beta+\delta} + \alpha_1 q_2 s_{\beta+\delta}$$

$$y/r = (1-\alpha_1 q_1) s_{\beta+\delta} - \alpha_1 q_2 c_{\beta+\delta}, \qquad z/r = q_1 \tag{33}$$

Perturbing accelerations due to radiation pressure can be obtained using Aksnes,[25] drag acceleration using Refs. 15 and 26, and small impulsive ΔV maneuvers using Ref. 19 (p. 456).

VI. Method of Multiple Scale

In this section, the method of multiple scale[13] is applied to Eq. (12). According to this method, the column matrix

$$\zeta = [h \; e_r \; e_\theta \; q_1 \; q_2 \; q_3 \; \delta t]^T \tag{34}$$

is assumed to be a function of a sequence of independent angle scales, a fast angle $\beta_0 = \beta$, and slow angles, $\beta_i = \epsilon^i \beta$ where i is positive integer and ϵ a small parameter of the order of the perturbing acceleration. To demonstrate this method, a first-order expansion is considered as follows

$$F = \epsilon F_1, \qquad \zeta = \zeta_0 + \epsilon \zeta_1$$

$$\frac{d}{d\beta} = D_0 + \epsilon D_1, \qquad D_i = \frac{\partial}{\partial \beta_i} \qquad (i = 0, 1) \tag{35}$$

Substitution of Eq. (34) into Eq. (12) and equating equal powers of ϵ on both sides of the resulting equation lead to

zeroth order:

$$D_0 h_0 = 0, \qquad D_0 e_{r0} = e_{\theta 0}, \qquad D_0 e_{\theta 0} = -e_{r0}$$

$$D_0 q_{10} = q_{20}, \qquad D_0 q_{20} = -q_{10}, \qquad D_0 q_{30} = 0, \qquad D_0 \delta_0 = 0$$

$$D_0 t_0 = (h_0^3/\mu_e^2) E_0^{-2}, \qquad E_0 = 1 + e_{r0} \tag{36}$$

first order:

$$D_0 h_1 = h_0 F_{\theta 1} - D_1 h_0$$

$$D_0 e_{r1} = e_{\theta 1} + 2E_0 F_{\theta 1} - D_1 e_{r0}$$

$$D_0 e_{\theta 1} = -e_{r1} - E_0 F_{r1} + e_{\theta 0} F_{\theta 1} - D_1 e_{\theta 0}$$

$$D_0 q_{11} = q_{21} - D_1 q_{10}$$

$$D_0 q_{21} = -q_{11} + q_{30} F_{h1} - D_1 q_{20}$$

$$D_0 q_{31} = -q_{20} F_{h1} - D_1 q_{30}$$

$$D_0 \delta_1 = \frac{q_{10}}{1+q_{30}} F_{h1} - D_1 \delta_0$$

$$D_0 t_1 = \frac{h_0^3}{\mu_e^2}\left[3\frac{h_1}{h_0} E_0^{-2} - 2e_{r1} E_0^{-3}\right] - D_1 t_0 \tag{37}$$

The solution of Eq. (36) is then

$$h_0 = h_0(\beta_1)$$

$$e_{r0} = e(\beta_1)\cos[\beta_0 + \phi_e(\beta_1)]$$

$$e_{\theta 0} = -e(\beta_1)\sin[\beta_0 + \phi_e(\beta_1)]$$

$$q_{10} = s(\beta_1)\sin[\beta_0 + \phi_q(\beta_1)]$$

$$q_{20} = s(\beta_1)\cos[\beta_0 + \phi_q(\beta_1)]$$

$$q_{30} = q_{30}(\beta_1)$$

$$t_0 = \tau(\beta_1) + \frac{h_0^3}{\mu_e^2}\int E_0^{-2}\,\partial\beta_0$$

$$\delta_0 = \delta_0(\beta_1) \tag{38}$$

The integral in the t_0 equation can be solved in closed form using an eccentric anomaly similar to that used in the well-known Kepler's time equation. In many cases, however, higher order solutions may require expression of t_0 in the form of power series of $e(\beta_1)$. This can be easily done by substitution of e_{r0} of Eq. (38) in E_0 of Eq. (36) followed by expansion of E_0^{-2} in power series of $e(\beta_1)$.

To obtain the first-order solution, Eq. (38) is substituted in Eq. (37) and the partial derivatives $\partial/\partial\beta_1[h_0, e, \phi_e, s_1, \phi_q, q_{30}, \delta_0, \tau]$ are used to eliminate the secular terms in Eq. (37).

Note that for Earth's zonal harmonics application, the perturbing acceleration (F_r, F_θ, F_n) are independent of δ and

t. In this case, the first six equations of Eq. (12) can be solved using the multiple scale and the remaining δ and t equations can then be solved separately.

Computer programs[27,28] are available to manipulate the formidable algebra that is involved in such perturbation analysis. Recently, Kamel and Duhamel[29] has applied the method of multiple scales to the main problem of artificial satellites using the REDUCE program.[27]

VII. Methods of Averaging

The Krylov-Bogoliubov method of averaging,[14] Lie transform,[15,16] and Lie series[17] are essentially related. In this section, the method of Lie transforms is selected to demonstrate the application of these methods to Eq. (19) to second order. First, the time equation t' will be expanded about an unperturbed trajectory specified by the initial conditions (h_0, e_{10}, e_{20}) or, more generally, by constant bias (h_b, e_{1b}, e_{2b}) as follows

$$h = h_b + \Delta h, \quad e_1 = e_{1b} + \Delta e_1, \quad e_2 = e_{2b} + \Delta e_2, \quad t = t_b + \Delta t \tag{39}$$

Substitution of Eq. (39) in the time equation t' of Eq. (19) leads to

$$t'_b = \frac{h_b^3}{\mu_e^2} E_b^{-2}$$

$$\Delta t' = \frac{h_b^3}{\mu_e^2} E_b^{-2}\left[3\frac{\Delta h}{h_b} - 2\Delta e_r E_b^{-1}\right] + \frac{3h_b^3}{\mu_e^2} E_b^{-2}\left[\left(\frac{\Delta h}{h_b}\right)^2 - 2\frac{\Delta h}{h_b}\Delta e_r E_b^{-1} + \Delta e_r^2 E_b^{-2}\right] + \ldots \tag{40}$$

where

$$E_b = 1 + e_{1b}\cos\beta + e_{2b}\sin\beta, \qquad \Delta e_r = \Delta e_1 \cos\beta + \Delta e_2 \sin\beta$$

Note that the t'_b equation can be solved in a closed form using eccentric anomaly as in Kepler's time equation. In many cases, however, it may be necessary to express t_b in the form of power series in (e_{1b}, e_{2b}). Also, the time equation $\Delta t'$ may not be solved unless the functions E^{-p} $(p = 1,2,...)$ are expanded as power series in (e_{1b}, e_{2b}). In these cases, the necessary expansions can be obtained in a similar manner as given by Eqs. (6) and (10) of Section 3.10 of Ref. 3 with e^n ($\cos nf$, $\sin nf$) replaced by

$$e^n \cos nf = A_n \cos n\beta - B_n \sin n\beta$$

$$e^n \sin nf = A_n \sin n\beta + B_n \cos n\beta \tag{41}$$

where

$$A_n = \text{Re}(e_{1b} - je_{2b})^n, \quad B_n = \text{Im}(e_{1b} - je_{2b})^n, \qquad j = \sqrt{-1}$$

The eccentricity e and the true anomaly f are as defined for Eq. (16).

Now, let y be the column matrix defined by

$$y = [\Delta h \ \Delta e_1 \ \Delta e_2 \ h_1 \ h_2 h_3 \ \delta \ \Delta t]^T \tag{42}$$

In view of Eqs. (19) and (40), we get

$$y' = \epsilon g_1(y,\beta) + (\epsilon^2/2) g_2(y,\beta) \tag{43}$$

In general, Eq. (43) cannot be solved exactly for arbitrary ϵ, but its solution is known for $\epsilon = 0$. This suggests the search for near-identity transformation $y \to x$ in the form of power series

$$y = x + \epsilon x_1(x,\beta) + (\epsilon^2/2) x_2(x,\beta) \tag{44}$$

so that the resulting system of differential equations

$$x' = \epsilon f_1(x) + (\epsilon^2/2) f_2(x) \tag{45}$$

will not contain the angle β or the time t. This new form will be more tractable than Eq. (43).

Using the procedure outlined in Refs. 15 and 16, we get

$$f_1(x) = g_1(x,\beta) - \frac{\partial W_1}{\partial \beta}$$

$$f_2(x) = g_2(x,\beta) - L_1[f_1(x) + g_1(x,\beta)] - \frac{\partial W_2}{\partial \beta} \tag{46}$$

where

$$L_1 F = TW_1 - GF$$

T = matrix whose ij elements are $\dfrac{\partial F_i}{\partial x_j}$

G = matrix whose ij elements are $\dfrac{\partial W_{1i}}{\partial x_j}$

(F_i, W_{1i}) = the ith element of the column matrices F and W_1, respectively

x_j = the jth element of the column matrix x

Now, the generating column matrices W_1 and W_2 are selected to eliminate the short-period terms (terms depending on β or t) in Eq. (46). Once this is done, the transformation [Eq. (44)] can be obtained from

$$x_1 = -W_1, \qquad x_2 = -W_2 + GW_1 \tag{47}$$

Note that if the perturbing acceleration is not a function of time (e.g., in case of Earth's zonal harmonics), the Lie transform of Eq. (46) can be applied to the first six equations of Eq. (19). The remaining $(\delta, \Delta t)$ equations are then solved separately.

VIII. Conclusions

Two new forms of the perturbed satellite equations of motion have been developed in terms of nonsingular orbital elements and perturbing acceleration components in the Euler-Hill rotating frame. The ambiguity in the application to retrograde orbits has been eliminated at the cost of adding the cosine of the orbital inclination to the satellite state vector. This addition results in a trivial integral of motion that may be used as a measure of the numerical integration stability and/or accuracy. For general perturbation applications, the addition of the cosine of the orbital inclination eliminates the manipulation of functions under square roots.

Transformation matrix between the Cartesian and the Euler-Hill frames is given in terms of the Cartesian as well as the state vectors used in the equations of motion. This matrix is useful in the evaluation of the initial condition for integration and also to compute the satellite position and velocity in Cartesian components once the equations are integrated.

Perturbing acceleration components due to Earth's zonal and tesseral harmonics and a third body have been developed in a form suitable for the integration of the equations of motion, either numerically or analytically.

The method of multiple scale to first order and the method of Lie transform to second order are outlined to demonstrate the use of the developed equations in connection with the general perturbation theories.

References

[1] Herrick, S., *Astrodynamics,* Vol. 1, Van Nostrand Reinhold Co., London, 1971, pp. 279-310.

[2] Tanaka, T. and Kawase, S., "A High Speed Integration Method for the Satellite's Ephemeris Generation," *Journal of Guidance and Control,* Vol. 1, May-June 1978, pp. 222-223.

[3] Smart, M.A., *Celestial Mechanics,* John Wiley & Sons, New York, 1953, pp. 26-78.

[4] Brouwer, D. and Clemence, G., *Methods of Celestial Mechanics,* Academic Press, New York, 1961, pp. 530-595.

[5] Deprit, A., "Ideal Frames for Perturbed Keplerian Motions," *Celestial Mechanics,* Vol. 13, March 1976, pp. 253-263.

[6] Cefola, P., "Equinoctial Orbit Elements; Application to Artificial Satellite Orbits," AIAA Paper 72-937, Sept. 1972.

[7] Nacozy, P. and Dallas, S., "The Geopotential in Nonsingular Orbital Elements," *Celestial Mechanics,* Vol. 15, Aug. 1977, pp. 453-466.

[8] Stiefel, E. and Scheifele, G., "Linear and Regular Celestial Mechanics," Springer-Verlag, Heidelberg, 1971, pp. 100-110.

[9] Broucke, R. and Lass, H., "On Redundant Variables in Lagrangian Mechanics With Applications to Perturbation Theory and KS Regularization," *Celestial Mechanics,* Vol. 12, No. 3, 1975, pp. 317-325.

[10] Deprit, A., "The Elimination of the Parallax in Satellite Theory," *Celestial Mechanics,* Vol. 24, No. 2, 1981, pp. 111-153.

[11] Struble, R.A., *Nonlinear Differential Equations,* McGraw-Hill Book Co., New York, 1962, pp. 70-72.

[12] Kevorkian, J., "The Two Variable Expansion Procedure for the Approximate Solution of Certain Nonlinear Differential Equations," *Lectures in Applied Mathematics,* Vol. 7, American Mathematical Society, Providence, R.I., 1966, p. 206.

[13] Nayfeh, A.H., *Perturbation Methods,* John Wiley & Sons, New York, 1973.

[14] Krylov, N. and Bogoliubov, N.N., "Introduction to Nonlinear Mechanics," *Annals of Mathematical Studies,* Vol. II, Princeton University Press, Princeton, N.J., 1947, pp. 100-110.

[15] Kamel, A.A., "Perturbation Method in the Theory of Nonlinear Oscillations," NASA CR-1554, March 1970; also *Celestial Mechanics,* Vol. 3, Dec. 1970, pp. 90-106.

[16] Kamel, A.A., "Lie Transforms and the Hamiltonization of Non-Hamiltonian Systems," *Celestial Mechanics,* Vol. 4, Dec. 1971, pp. 397-405.

[17] Hori, G., "Theory of General Perturbations for Noncanonical Systems," *Publications of the Astronomical Society of Japan,* Vol. 23, Jan. 1971, pp. 567-587.

[18] Bate, R.R., Mueller, D.D., and White, J.E., *Fundamentals of Astrodynamics,* Dover Publications Inc., New York, 1971, pp. 17-25.

[19] Meirovitch, L., *Methods of Analytical Dynamics,* McGraw-Hill Book Co., New York, 1970, pp. 140-143, 447-466.

[20] Pines, S., "Uniform Representation of the Gravitational Potential and its Derivatives," *AIAA Journal,* Vol. 11, Nov. 1973, pp. 1508-1511.

[21] Broucke, R., "On the Expansion of the Gravitational Potential with Automated Poisson Series Manipulation," AIAA Paper 76-814, Aug. 1976.

[22] Brown, E., *An Introductory Treatise on the Lunar Theory,* Cambridge University Press, New York, 1896, p. 79.

[23] Broucke, R., "Expansion of the Third-Body Disturbing Function," *Journal of Guidance and Control,* Vol. 4, May-June 1981, pp. 346-347.

[24] Doggett, L.E., Kaplan, G.H., and Seidelmann, P.K., *Almanac for Computers,* Nautical Almanac Office, U.S. Naval Observatory, Washington, D.C., 1978.

[25] Aksnes, K., "Short-Period and Long-Period Perturbations of a Spherical Satellite Due to Direct Solar Radiation," *Celestial Mechanics,* Vol. 13, No. 1, 1976, pp. 89-104.

[26] Jacchhia, L.G., "Thermospheric Temperature, Density and Composition: New Models," Smithsonian Astrophysical Observatory, Spec. Rept. 375, March 1977.

[27] Hearn, A.C., "REDUCE Manual," Dept. of Computer Science, University of Utah, Salt Lake City, March 1973.

[28] Dasenbrock, R.R., "Algebraic Manipulation by Computers," Naval Research Laboratory, Washington, D.C., Rept. 73-1, June 29, 1973.

[29] Kamel, A. and Duhamel, T., "A Second Order Solution of the Main Problem of Artificial Satellites using Multiple Scales," AIAA Paper 82-1408, Aug. 1982.